THE HAMILTONICITY OF BLOCK-INTERSECTION GRAPHS OF BALANCED INCOMPLETE BLOCK DESIGNS

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The Hamiltonicity of Block-Intersection Graphs of Balanced Incomplete Block Designs.

by

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Abstract

Given a balanced incomplete block design $\mathcal{D} = (V, \mathcal{B})$ with block set $\mathcal{B}$, its traditional *block-intersection graph* $G(\mathcal{D})$ is the graph having vertex set $\mathcal{B}$ such that two vertices $\beta_1, \beta_2 \in \mathcal{B}$ are adjacent if and only if $\beta_1 \cap \beta_2 \neq \emptyset$. The *I-block-intersection graph* of a design $\mathcal{D} = (V, \mathcal{B})$, denoted by $G_I(\mathcal{D})$, is the graph having vertex set $\mathcal{B}$ such that two vertices $\beta_1, \beta_2 \in \mathcal{B}$ are adjacent if and only if $|\beta_1 \cap \beta_2| \in I$, where I is a given subset of $\{1, 2, \ldots, k\}$. If $|I| = 1$ then we will also refer to the I-block-intersection graph as the i-block-intersection graph and will denote it by $G_i(\mathcal{D})$, where i is the sole element of I.

The initial investigation into the cycle properties of block-intersection graphs was said to have been initiated by Ron Graham in 1987. One year later, Graham's question was proved by Peter Horák and Alexander Rosa. Since the posing of Graham's question, many people have looked into several different cycle properties of block-intersection graphs, most of which can be found in [1, 4, 6, 9, 10, 13–15].

In this thesis we will prove several lemmata that deal with the size of independent sets of vertices in block-intersection graphs. Also, we will show that the $\{1, 2\}$-block-intersection graph of any

1. $(v, 4, \lambda)$-BIBD with arbitrary λ is Hamiltonian for $v \geq 11$,

2. $(v, 5, \lambda)$-BIBD with arbitrary λ is Hamiltonian for $v \geq 57$,

3. $(v, 6, \lambda)$-BIBD with arbitrary λ is Hamiltonian for $v \geq 167$.

We then extend the idea of Horák, Pike and Raines [11], and prove that the 1-block-intersection graph of any $(v, 4, \lambda)$-BIBD with arbitrary λ is Hamiltonian for $v \geq 136$. Finally we end with some open problems related to extensions of work done in this thesis.

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Chapter 1

Introduction

Hamiltonian cycles in graphs have been one of the central topics in graph theory. Determining whether or not a graph is Hamiltonian is an NP-complete problem. Hence it seems to be impossible to find an efficient standard for determining when a graph is Hamiltonian.

The motivation of this thesis is due to Dr. David Pike, who is one of the leading experts in the study of block-intersection graphs, and together with Peter Horák and Michael Raines, has shown that the 1-block-intersection graph of any $(v, 3, \lambda)$-BIBD with $v \geq 12$ and arbitrary λ is Hamiltonian [11].

At a regional meeting of the American Mathematical Society in March 1987, Ron Graham asked if the block-intersection graph of every $(v, 3, 1)$-BIBD is Hamiltonian

[2]. This is said to have been the initial motivation into investigating the cycle properties of block-intersection graphs. In 1988, one year after Graham posed this question, the question was proved by Horák and Rosa [12]. Over the next two decades many people have looked into several different cycle properties of block-intersection graphs, most of which can be found in [1, 4, 6, 9, 10, 13–15].

In this thesis we will investigate block-intersection graphs. More specifically, Sections one and two of Chapter 1 will cover background material in graph theory and design theory, respectively. Chapter 2 will present some lemmata bounding the sizes of independent sets of vertices in block-intersection graphs, as well as notation that will be used throughout the rest of the thesis. Chapter 3 shows that the $\{1,2\}$-block-intersection graph of any

1. $(v, 4, \lambda)$-BIBD is Hamiltonian for $v \geq 11$ and arbitrary λ,
2. $(v, 5, \lambda)$-BIBD is Hamiltonian for $v \geq 57$ and arbitrary λ,
3. $(v, 6, \lambda)$-BIBD is Hamiltonian for $v \geq 167$ and arbitrary λ.

This is original work following ideas used in [7, 11, 12]. Chapter 4 shows that the 1-block-intersection graph of a $(v, 4, \lambda)$-BIBD is Hamiltonian for $v \geq 136$ and arbitrary λ. This is an extension of the work done in [11]. Chapter 5 poses some open problems related to extensions of work done in this thesis.

1.1 Graph Theory

This section contains some preliminary definitions, examples and theorems from graph theory, which are relevant to the development of the topics and results presented in this thesis.

A *graph* G is a pair (V, E) of sets, V nonempty and each element of E a set of two distinct elements of V. The elements of V are called *vertices*; the elements of E are called *edges*. An example of a graph is given in Figure 1.1, in which $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $E = \{v_1v_2, v_1v_5, v_1v_7, v_2v_7, v_2v_5, v_5v_7, v_2v_3, v_5v_6, v_3v_4, v_4v_6\}$.

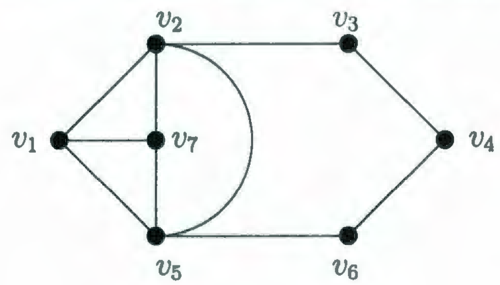

Figure 1.1: G

A graph G' is a *subgraph* of another graph G if and only if the vertex and edge sets of G' are, respectively, subsets of the vertex and edge sets of G. Figures 1.2 and 1.3 are both subgraphs of Figure 1.1.

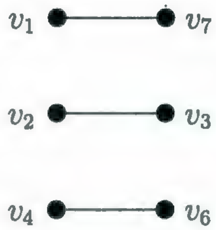

Figure 1.2: G_1'

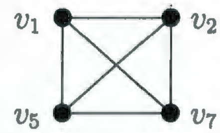

Figure 1.3: G_2'

The *complete graph* on n vertices, denoted by K_n, is the unique graph up to isomorphism on n vertices with an edge between every pair of vertices. The complete graph on seven vertices is given in Figure 1.4.

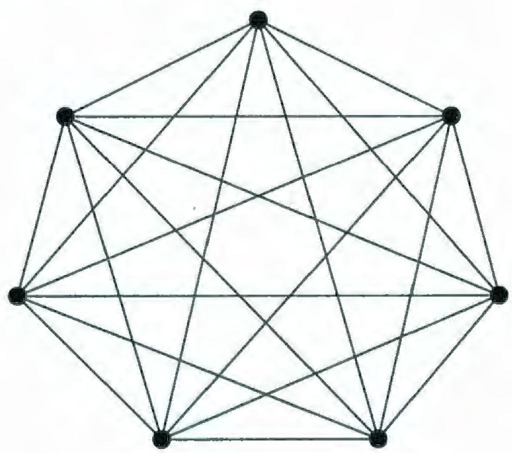

Figure 1.4: K_7

A *walk* is an alternating sequence of vertices and edges, beginning and terminating at vertices, with each edge being incident to the vertices immediately preceding and succeeding it in the sequence. A walk is *closed* if and only if the first vertex is the same as the last; otherwise the walk is *open*. Figure 1.5 provides an example of a closed walk.

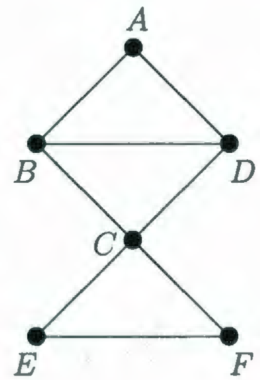

Figure 1.5: Closed Walk: $CEFCBDC$

A *cycle* is a closed walk with no repeated vertices. A *Hamiltonian cycle* in a graph is a cycle in which every vertex of the graph appears. Figure 1.6 provides an example of a Hamiltonian cycle.

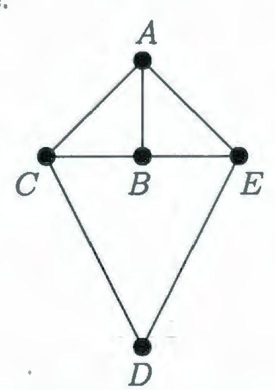

Figure 1.6: Hamiltonian Cycle: $ABCDEA$

A *vertex-cut* or *cutset* of a graph G is a set $S \subset V(G)$ such that $G - S$ has more than one component. A *component* of a graph is a maximal connected subgraph, that is, a connected subgraph which is properly contained in no other connected subgraph which has more vertices or more edges. Figure 1.7 is an example of a graph with more than one component.

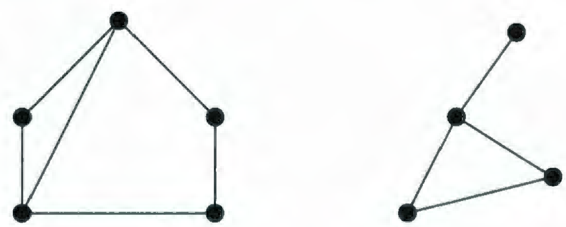

Figure 1.7: A graph with two components

A graph G is k*-connected* if every vertex-cut has at least k vertices. The *connectivity* of G, denoted by $\kappa(G)$, is the maximum k such that G is k-connected. In Figures 1.8 and 1.9 the connectivity is $\kappa = 1$ and $\kappa = 3$, respectively.

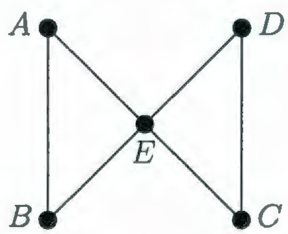

Figure 1.8: $\kappa = 1$

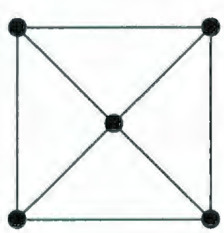

Figure 1.9: $\kappa = 3$

A set of vertices is *independent* if no two vertices in the set have an edge between them. A maximum independent set is a largest independent set for a given graph G and its size is denoted by $\alpha(G)$. In Figure 1.8, the vertices in the set $\{A, D\}$ are independent, as are those in $\{B, C\}$.

In 1972, Vašek Chvátal and Paul Erdős gave the following sufficient condition for a graph to be Hamiltonian.

Theorem 1.1. *[7]* For any graph G, if $\alpha(G) \leq \kappa(G)$, then G is Hamiltonian.

Theorem 1.1 has opened a whole new world in the study of the Hamiltonicity of graphs. Since [7], several classes of graphs have been shown to be Hamiltonian, many of which establish their conclusions by using Theorem 1.1.

1.2 Design Theory

This section contains some preliminary definitions, examples and theorems of design theory that are relevant to the development of the topics and results presented in this thesis.

A *balanced incomplete block design* (v, b, r, k, λ)-BIBD, is a pair $(V, \mathcal{B})$ where V is a set of v elements and $\mathcal{B}$ is a collection of k-subsets of V called *blocks* (with repetition of blocks allowed) where any 2-subset of V is contained in exactly λ blocks. We write (v, k, λ)-BIBD for short since the parameters b and r can be determined from knowing v, k and λ.

The five parameters of a (v, k, λ)-BIBD are as follows:

1. v, the size of V,
2. b, the size of $\mathcal{B}$ (i.e., the number of blocks),
3. r, the number of blocks in which an element appears (this is the same for all elements and is often referred to as the *replication number*),
4. k, the size of the blocks (incompleteness dictates that $k < v$),
5. λ, the *index* of the design, is the number of times a pair of elements of V appears in the design. Balance ensures that this is the same for all pairs of elements.

Example 1.1. Consider the blocks $\beta_1 = \{1,2,4\}, \beta_2 = \{2,3,5\}, \beta_3 = \{3,4,6\}, \beta_4 = \{4,5,7\}, \beta_5 = \{1,5,6\}, \beta_6 = \{2,6,7\}, \beta_7 = \{1,3,7\}$. These blocks form the $(7,3,1)$-BIBD. Since the blocks satisfy $|\beta_i \cap \beta_j| \neq \emptyset$ for all $i \neq j$, the block-intersection graph of the $(7,3,1)$-BIBD is K_7, as portrayed in Figure 1.4.

The following are two conditions which allow us to solve for all parameters of a BIBD when only knowing v, k, and λ.

Lemma 1.1. *(see [3])* In a (v,k,λ)-BIBD with b blocks each element occurs in r blocks where

1. $\lambda(v-1) = r(k-1)$,

2. $bk = rv$.

Proof. 1. Consider an element $x \in V$ and let $P = |\{(y, B) \mid B \in \mathcal{B}, x, y \in B \text{ and } x \neq y\}|$. Since x forms λ pairs with each of the remaining $v - 1$ elements, $P = \lambda(v - 1)$. As well, x occurs in r blocks, each of which contain $k - 1$ additional elements, so $P = r(k - 1)$. Hence $\lambda(v - 1) = r(k - 1)$.

2. Let $T = |\{(x, B) \mid x \in V, B \in \mathcal{B}, x \in B\}|$. Since there are k elements in each of the b blocks, $T = bk$. As well, each of the v elements occurs exactly r times, so $T = vr$. Hence $bk = rv$.

□

Chapter 2

Block-Intersection Graphs

In this chapter we will present some lemmata for bounding the size of independent sets of vertices in block-intersection graphs, as well as notation that will be used throughout the rest of the thesis.

Given a balanced incomplete block design $\mathcal{D} = (V, \mathcal{B})$ with block set $\mathcal{B}$, its traditional *block-intersection graph* $G(\mathcal{D})$ is the graph having vertex set $\mathcal{B}$ such that two vertices $\beta_1, \beta_2 \in \mathcal{B}$ are adjacent if and only if $\beta_1 \cap \beta_2 \neq \emptyset$. The *$I$-block-intersection graph* of a design $\mathcal{D} = (V, \mathcal{B})$, denoted by $G_I(\mathcal{D})$, is the graph having vertex set $\mathcal{B}$ such that two vertices $\beta_1, \beta_2 \in \mathcal{B}$ are adjacent if and only if $|\beta_1 \cap \beta_2| \in I$, where I is a given subset of $\{1, 2, \ldots, k\}$. If $|I| = 1$ then we will also refer to the I-block-intersection graph as the i-block-intersection graph and will denote it by $G_i(\mathcal{D})$, where i is the sole element of I.

Theorems 3.1, 3.2, 3.3 and 4.1 proved in this thesis follow a similar proof technique used in [11]. First note that these theorems have already been proven for $\lambda = 1$ [12]. Our goal is to establish that the Chvátal-Erdős condition (namely the hypothesis of Theorem 1.1) holds true throughout the proof of these theorems.

We define the following notation to be used throughout the proofs of the theorems to be presented.

1. For sets $S_1, S_2, \ldots, S_n$, the Inclusion-Exclusion Principle states

$$\left|\bigcup_{i=1}^{n} S_i\right| = \sum_{i=1}^{n} |S_i| - \sum_{\substack{i,j \\ 1\leq i<j\leq n}} |S_i \cap S_j| + \sum_{\substack{i,j,k \\ 1\leq i<j<k\leq n}} |S_i \cap S_j \cap S_k| - \cdots + (-1)^{n-1}|S_1 \cap \cdots \cap S_n|.$$

2. To represent the number of blocks of $\mathcal{B}$ having at least i elements in common with a set U we calculate

$$\sum_{j=i}^{k} \left(A(i,j)(-1)^{j-i} \sum_{\substack{T\subseteq U \\ |T|=j}} \nu(T) \right),$$

where the $A(i,j)$ are numerical constants that are suitably chosen to ensure that each block to be counted is counted exactly once, and $\nu(T)$ represents the number of blocks of $\mathcal{B}$ that contain T. For example, if $|T| = 1$ then $\nu(T) = r$. If $|T| = 2$ then $\nu(T) = \lambda$.

To illustrate how the constants $A(i,j)$ are determined, suppose that $i = 3$ and consider a set $U = \{u, w, x, y, z\}$ as an example. If there is a block that contains exactly three points of U then the $j = 4$ and $j = 5$ parts of the sum count this block zero times, and so it must be that $A(3,3) = 1$.

If there is a block that contains exactly four points of U, then the $j = 3$ part of the sum counts this block $A(3,3)\binom{4}{3}$ times, which is three more than is wanted. Therefore $A(3,4) = 3$.

If there is a block that contains all five points of U, then the $j = 3$ part of the sum counts this block $A(3,3)\binom{5}{3}$ times and the $j = 4$ part of the sum counts this block $-A(3,4)\binom{5}{4}$ times. To obtain a net count of exactly one it must be that $A(3,5) = 6$.

Definition 2.1. *(see [8])*

$$\binom{n}{k} = \begin{cases} \frac{n^{\underline{k}}}{k^{\underline{k}}} & , \quad k \geq 0 \\ 0 & , \quad \text{otherwise} \end{cases},$$

where $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$.

A consequence of the above definition gives the following property called *upper negation.*

Corollary 2.1.

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k}.$$

Proof.

$$
\begin{aligned}
\binom{n}{k} &= \frac{n^{\underline{k}}}{k!} \\
&= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \\
&= (-1)^k \cdot \frac{(k-n-1)(k-n-2)\cdots(1-n)(-n)}{k!} \\
&= (-1)^k \cdot \frac{(k-n-1)(k-n-2)\cdots(k-n-1-k+1)}{k!} \\
&= (-1)^k \frac{(k-n-1)^{\underline{k}}}{k!} \\
&= (-1)^k \binom{k-n-1}{k}.
\end{aligned}
$$

□

Lemma 2.1. $A(i,j) = \binom{j-1}{j-i}$, where $i \in \{1, 2, \ldots, k\}$.

Proof. We proceed with a proof by induction. Let $i \leq j \leq k$ and $i \in \{1, 2, \ldots, k\}$. For $j = i$, then $A(i,i) = 1 = \binom{j-1}{j-i}$, as our coefficient $A(i,j)$ is always one for our smallest i value. Assume $A(i,j) = \binom{j-1}{j-i}$ for all $j < t$, where $i < t \leq k$. We want to prove $A(i,j) = \binom{j-1}{j-i}$ for $j = t$. We consider two cases.

Case 1: If $t \equiv i (\mathrm{mod}\ 2)$ we need

$$A(i,i)\binom{t}{i} - A(i,i+1)\binom{t}{i+1} + \cdots + A(i,t)\binom{t}{t} = 1$$

or equivalently

$$A(i,t) = 1 - \left(A(i,i)\binom{t}{i} - A(i,i+1)\binom{t}{i+1} + \cdots - A(i,t-1)\binom{t}{t-1}\right),$$

because as we said before we want to make sure that each block to be counted is counted exactly once. Hence

$$\begin{aligned}
A(i,t) &= 1-\sum_{x=i}^{t-1}(-1)^{x-i}\binom{x-1}{x-i}\binom{t}{x} \\
&= 1-\sum_{x}(-1)^{x-i}\binom{x-1}{x-i}\binom{t}{x}+(-1)^{t-i}\binom{t-1}{t-i}\binom{t}{t} \\
&= 1+\binom{t-1}{t-i}-\sum_{x}((-1)^{x-i})^2\binom{t}{x}\binom{x-i-(x-1)-1}{x-i} \\
&= 1+\binom{t-1}{t-i}-\sum_{x}\binom{t}{x}\binom{-i}{x-i} \\
&= 1+\binom{t-1}{t-i}-\left(\binom{t}{0}\binom{-i}{-i}+\sum_{x\neq 0}\binom{t}{x}\binom{-i}{-x}\right) \\
&= 1+\binom{t-1}{t-i}-1 \\
&= \binom{t-1}{t-i},
\end{aligned}$$

Case 2: If $t \not\equiv i \pmod 2$, then similar to Case 1

$$A(i,i)\binom{t}{i}-A(i,i+1)\binom{t}{i+1}+A(i,i+2)\binom{t}{i+2}-\cdots-A(i,t)\binom{t}{t}=1.$$

Hence

$$\begin{aligned}
A(i,t) &= -1+\sum_{x=i}^{t-1}(-1)^{x-i}\binom{x-1}{x-i}\binom{t}{x} \\
&= -1+\sum_{x}(-1)^{x-i}\binom{x-1}{x-i}\binom{t}{x}-(-1)^{t-i}\binom{t-1}{t-i}\binom{t}{t} \\
&= -1+\binom{t-1}{t-i}+\sum_{x}((-1)^{x-i})^2\binom{t}{x}\binom{x-i-(x-1)-1}{x-i} \\
&= -1+\binom{t-1}{t-i}+\sum_{x}\binom{t}{x}\binom{-i}{x-i} \\
&= -1+\binom{t-1}{t-i}+\left(\binom{t}{0}\binom{-i}{-i}+\sum_{x\neq 0}\binom{t}{x}\binom{-i}{-x}\right) \\
&= -1+\binom{t-1}{t-i}+1 \\
&= \binom{t-1}{t-i}.
\end{aligned}$$

So $A(i,j)=\binom{j-1}{j-i}$ for $j=t$ and we are done.

□

Lemma 2.2. The size of an independent set I of vertices in the $\{1,2\}$-block-intersection graph of a (v,k,λ)-BIBD is bounded above by $\frac{v((\lambda-1)(k-1)+2)}{2k}$.

Proof. Choose an independent set of vertices $I \subset \mathcal{B}$ of $G_{\{1,2\}}(\mathcal{D})$ and a point $a_1 \in V$. Define I_{a_1} to be the set of blocks in I that contain the point a_1 and let $S = \{a_1, a_2, \ldots, a_k\} \in I_{a_1}$ be a block of size k. Since the blocks of I_{a_1} must each have at least three points in common with S, one of which is a_1, then each block of I_{a_1} contains at least two pairs of the form $\{a_1, a_i\}$ where $2 \le i \le k$. Each pair of points occurs in λ blocks of $\mathcal{B}$ and so by counting pairs of points occurring in the blocks in I_{a_1} having the form $\{a_1, a_i\}$, where $i \in \{2, 3, \ldots, k\}$, we find that $2(|I_{a_1}| - 1) + k - 1 \leq (k-1)\lambda$. Hence $2(|I_p| - 1) + k - 1 \leq (k-1)\lambda$ for all $p \in V$, and therefore

$$|I| = \frac{1}{k}\sum_{p \in V} |I_p| \leq \frac{1}{k}\sum_{p \in V} \frac{(\lambda - 1)(k-1) + 2}{2} = \frac{v((\lambda-1)(k-1)+2)}{2k}.$$

$\square$

Lemma 2.3. The size of an independent set I of vertices in the $\{1, 2, \ldots, k-2\}$-block-intersection graph of a (v, k, λ)-BIBD is bounded above by $\frac{\lambda v}{k-1}$.

Proof. Choose an independent set of vertices $I \subset \mathcal{B}$ of $G_{\{1,2,\ldots,k-2\}}(\mathcal{D})$ and a point $a_1 \in V$. Define I_{a_1} to be the set of blocks in I that contain the point a_1 and let $S = \{a_1, a_2, \ldots, a_k\} \in I_{a_1}$ be a block of size k. If all blocks of I_{a_1} have all k points in common, then $|I_{a_1}| \leq \lambda$. Otherwise there are blocks in I_{a_1} which share only $k-1$ points with S. For $2 \leq i \leq k$, let $\mathcal{J}_{a_i} = \{\beta \in I_{a_1} \mid a_i \notin \beta\}$ denote the subset of I_{a_1} that consists of all blocks of I not containing a_i, but containing each point of $S - \{a_i\}$. If there is a unique $i \in \{2, 3, \ldots, k\}$ for which $\mathcal{J}_{a_i} \neq \emptyset$, then clearly $|I_{a_1}| \leq \lambda$. However, if $\mathcal{J}_{a_i} \neq \emptyset$ for multiple choices of $i \in \{2, 3, \ldots, k\}$ then there must be a point $a_{k+1} \in V - S$ that is shared by each block of $\mathcal{J}_{a_2} \cup \mathcal{J}_{a_3} \cup \cdots \cup \mathcal{J}_{a_k}$. Hence the pair $\{a_1, a_{k+1}\}$ occurs in each of these blocks and so $\sum_{i=2}^{k} |\mathcal{J}_{a_i}| = |\mathcal{J}_{a_2} \cup \mathcal{J}_{a_3} \cup \cdots \cup \mathcal{J}_{a_k}| \leq \lambda$.

Also, for each $i \in \{2, 3, \dots, k\}$ the pair $\{a_1, a_i\}$ occurs in each block of $I_{a_1} - \mathcal{J}_{a_i}$ and so $|I_{a_1}| - |\mathcal{J}_{a_i}| = |I_{a_1} - \mathcal{J}_{a_i}| \le \lambda$. It now follows that $(k-1)|I_{a_1}| \le k\lambda$. Hence $(k-1)|I_p| \le k\lambda$ for all $p \in V$, and therefore

$$|I| = \frac{1}{k}\sum_{p\in V} |I_p| \le \frac{1}{k}\sum_{p\in V} \frac{k\lambda}{k-1} = \frac{\lambda v}{k-1}.$$

□

Lemma 2.4. The size of an independent set I of vertices in the 1-block-intersection graph of a (v, k, λ)-BIBD is bounded above by $\frac{v((\lambda-1)(k-1)+1)}{k}$.

Proof. Choose an independent set of vertices $I \subset \mathcal{B}$ of $G_1(\mathcal{D})$ and a point $a_1 \in V$. Define I_{a_1} to be the set of blocks in I that contain the point a_1 and let $S = \{a_1, a_2, \dots, a_k\} \in I_{a_1}$ be a block of size k. Since the blocks of I_{a_1} must each have at least two points in common with S, one of which is a_1, then each block of I_{a_1} contains at least one pair of the form $\{a_1, a_i\}$ where $2 \le i \le k$. Each pair of points occurs in λ blocks of $\mathcal{B}$ and so by counting pairs of points in I_{a_1} having the form $\{a_1, a_i\}$, where $i \in \{2, 3, \dots, k\}$, we find that $(|I_{a_1}| - 1) + k - 1 \le (k-1)\lambda$. Hence $(|I_p| - 1) + k - 1 \le (k-1)\lambda$ for all $p \in V$, and therefore

$$|I| = \frac{1}{k}\sum_{p\in V} |I_p| \le \frac{1}{k}\sum_{p\in V} ((\lambda-1)(k-1)+1) = \frac{v((\lambda-1)(k-1)+1)}{k}.$$

□

Chapter 3

The $\{1, 2\}$-Block-Intersection Graph

In this chapter we will prove that the $\{1, 2\}$-block-intersection graph of any

1. $(v, 4, \lambda)$-BIBD with arbitrary λ is Hamiltonian for $v \geq 11$,
2. $(v, 5, \lambda)$-BIBD with arbitrary λ is Hamiltonian for $v \geq 57$,
3. $(v, 6, \lambda)$-BIBD with arbitrary λ is Hamiltonian for $v \geq 167$.

Theorem 3.1. The $\{1, 2\}$-block-intersection graph of a $(v, 4, \lambda)$-BIBD with $v \geq 11$ and arbitrary λ is Hamiltonian.

Proof. Let $\mathcal{D} = (V, \mathcal{B})$ be a $(v, 4, \lambda)$-BIBD and $G_{\{1,2\}}(\mathcal{D}) = (\mathcal{B}, E)$ denote the $\{1,2\}$-block-intersection graph of $\mathcal{D}$. We will show that $G_{\{1,2\}}(\mathcal{D})$ is Hamiltonian by showing that $\alpha(G_{\{1,2\}}(\mathcal{D})) \leq \kappa(G_{\{1,2\}}(\mathcal{D}))$.

First choose an independent set of vertices $I \subset \mathcal{B}$ of $G_{\{1,2\}}(\mathcal{D})$. Then by Lemma 2.3 we have $|I| \leq \frac{\lambda v}{3}$. Hence $\alpha(G_{\{1,2\}}(\mathcal{D})) \leq \frac{\lambda v}{3}$.

Now let $C \subset \mathcal{B}$ be a cutset of $G_{\{1,2\}}(\mathcal{D})$. We wish to show that $|C| \geq \frac{\lambda v}{3}$. Let A be the vertex set of a component of $G_{\{1,2\}}(\mathcal{D}) - C$. Then let $F = \mathcal{B} - (A \cup C)$, $V_A = \bigcup\limits_{\beta \in A} \beta$ and $V_F = \bigcup\limits_{\beta \in F} \beta$.

We now consider two cases.

Case 1: $V_A \cup V_F = V$.

Case 1a: Assume first that $V_A \cap V_F \neq \emptyset$. Then there is a block $S = \{w, x, y, z\} \in A$ and a block $S' = \{w, x, y, z'\} \in F$ having at least three points in common, say w, x, and y.

Case 1a)i: Suppose $z = z'$. From the inclusion-exclusion principle there are $4r - 6\lambda + \sum\limits_{\substack{T \subseteq S \\ |T|=3}} \nu(T) - \nu(S)$ blocks of $\mathcal{B}$ containing at least one point of S. Also, there are $\sum\limits_{\substack{T \subseteq S \\ |T|=3}} \nu(T) - 3\nu(S)$ blocks of $\mathcal{B}$ having at least three points in common with S. Hence there are $(4r - 6\lambda + \sum\limits_{\substack{T \subseteq S \\ |T|=3}} \nu(T) - \nu(S)) - (\sum\limits_{\substack{T \subseteq S \\ |T|=3}} \nu(T) - 3\nu(S))$ blocks with exactly one or exactly two points in common with S. Therefore $4r - 6\lambda + 2\nu(S)$ is a lower bound on the size of the cutset C and $|C| \geq 4r - 6\lambda + 2\nu(S) > 4\lambda\left(\frac{v-1}{3}\right) - 6\lambda \geq \frac{\lambda v}{3}$

for $v \geq 8$, since $\nu(S) \geq 2$.

Case 1a)ii: Suppose that $z \neq z'$. Then there are $3r - 3\lambda + \nu(\{w,x,y\})$ blocks of $\mathcal{B}$ containing at least one point of $\{w,x,y\}$. We wish to count the number of blocks containing exactly one point of $\{w,x,y\}$, ignoring whether or not these blocks will also contain z, z' or both, as these blocks will still be adjacent to both S and S'. Hence there are $(3r-3\lambda+\nu(\{w,x,y\})) - (3(\lambda-\nu(\{w,x,y\}))+\nu(\{w,x,y\}))$ blocks of $\mathcal{B}$ containing exactly one point of $\{w,x,y\}$, where $3(\lambda - \nu(\{w,x,y\}))$ is the number of blocks containing exactly two points of $\{w,x,y\}$ and $\nu(\{w,x,y\})$ is the number of blocks containing all the points of $\{w,x,y\}$. Therefore $|C| \geq 3r-6\lambda+3\nu(\{w,x,y\}) > 3\lambda\left(\frac{v-1}{3}\right) - 6\lambda \geq \frac{\lambda v}{3}$ for $v \geq 11$.

Case 1b: Assume $V_A \cap V_F = \emptyset$. That is, there is no block $S \in G_{\{1,2\}}(\mathcal{D}) - C$ satisfying $V_A \cap S \neq \emptyset \neq V_F \cap S$. Each block of $\mathcal{B}$ that contains a pair of points wx such that $w \in V_A$ and $x \in V_F$ must belong to C. Clearly there are $|V_A||V_F|$ such pairs of points. Since a block of size four can contain at most four pairs of this kind, and each pair of points occurs in λ blocks, $|C| \geq \frac{\lambda |V_A||V_F|}{4}$. Also, each component of $G_{\{1,2\}}(\mathcal{D}) - C$ has at least one block, so $|V_A| \geq 4$ and $|V_F| \geq 4$. Therefore $|C| \geq \min_{4 \leq p \leq v-4} \left\{\frac{\lambda p(v-p)}{4}\right\} \geq \lambda(v-4) \geq \frac{\lambda v}{3}$ for $v \geq 6$.

Case 2: Assume $V_A \cup V_F \subsetneq V$. Then all blocks containing any point of $V - (V_A \cup V_F)$ must be in C. If there are at least two points in $V - (V_A \cup V_F)$ then

$|C| \geq 2r - \lambda \geq \frac{\lambda v}{3}$ for $v \geq 5$.

Now suppose that $|V - (V_A \cup V_F)| = 1$ and let γ be the only element of $V - (V_A \cup V_F)$. So $|V_A \cup V_F| = v - 1$. If $V_A \cap V_F = \emptyset$, then similar to Case 1b, we obtain $|C| \geq \min_{4 \leq p \leq v-5} \left\{\frac{\lambda p(v-1-p)}{4}\right\} \geq \lambda(v-5) \geq \frac{\lambda v}{3}$ for $v \geq 8$.

If $V_A \cap V_F \neq \emptyset$, then there is a block $S = \{w, x, y, z\} \in A$, and a block $S' = \{w, x, y, z'\} \in F$ having at least three points in common, say w, x, and y. Then C includes all blocks containing exactly one point of $\{w, x, y\}$, of which there are exactly $(3r - 3\lambda + \nu(\{w, x, y\})) - (3\lambda - 2\nu(\{w, x, y\}))$. Hence $|C| \geq 3r - 6\lambda + 3\nu(\{w, x, y\}) > 3\lambda\left(\frac{v-1}{3}\right) - 6\lambda \geq \frac{\lambda v}{3}$ for $v \geq 11$.

Therefore $\kappa(G_{\{1,2\}}(\mathcal{D})) \geq \frac{\lambda v}{3}$ for all $v \geq 11$ and $\alpha(G_{\{1,2\}}(\mathcal{D})) \leq \kappa(G_{\{1,2\}}(\mathcal{D}))$ in all cases. Hence the Chvátal-Erdős condition holds, and so $G_{\{1,2\}}(\mathcal{D})$ is Hamiltonian.

□

Theorem 3.2. The $\{1,2\}$-block-intersection graph of a $(v, 5, \lambda)$-BIBD with $v \geq 57$ and arbitrary λ is Hamiltonian.

Proof. Let $\mathcal{D} = (V, \mathcal{B})$ be a $(v, 5, \lambda)$-BIBD and $G_{\{1,2\}}(\mathcal{D}) = (\mathcal{B}, E)$ denote the $\{1,2\}$-block-intersection graph of $\mathcal{D}$. We will show that $G_{\{1,2\}}(\mathcal{D})$ is Hamiltonian by showing that $\alpha(G_{\{1,2\}}(\mathcal{D})) \leq \kappa(G_{\{1,2\}}(\mathcal{D}))$.

First choose an independent set of vertices $I \subset \mathcal{B}$ of $G_{\{1,2\}}(\mathcal{D})$. Then by Lemma 2.2 we have $|I| \leq \frac{v(2\lambda-1)}{5}$. Hence $\alpha(G_{\{1,2\}}(\mathcal{D})) \leq \frac{v(2\lambda-1)}{5}$.

Now let $C \subset \mathcal{B}$ be a cutset of $G_{\{1,2\}}(\mathcal{D})$. We wish to show that $|C| \geq \frac{v(2\lambda-1)}{5}$. Let A be the vertex set of a component of $G_{\{1,2\}}(\mathcal{D}) - C$. Then let $F = \mathcal{B} - (A \cup C)$, $V_A = \bigcup_{\beta \in A} \beta$ and $V_F = \bigcup_{\beta \in F} \beta$.

We now consider two cases.

Case 1: $V_A \cup V_F = V$.

Case 1a: Assume first that $V_A \cap V_F \neq \emptyset$. Then there is a block $S = \{u, w, x, y, z\} \in A$ and a block $S' = \{u, w, x, y', z'\} \in F$ having at least three points in common, say u, w and x.

Case 1a)i: Suppose $|\{y, z\} \cap \{y', z'\}| = 2$. Without loss of generality, assume $y = y'$ and $z = z'$. From the inclusion-exclusion principle there are $5r - 10\lambda + \sum_{j=3}^{5}((-1)^{j-1} \sum_{\substack{T \subseteq S \\ |T|=j}} \nu(T))$ blocks of $\mathcal{B}$ containing at least one point of S. Also, there are $\sum_{\substack{T \subseteq S \\ |T|=3}} \nu(T) - 3\sum_{\substack{T \subseteq S \\ |T|=4}} \nu(T) + 6\nu(S)$ blocks of $\mathcal{B}$ containing at least three points of S. Hence there are $(5r - 10\lambda + \sum_{j=3}^{5}((-1)^{j-1} \sum_{\substack{T \subseteq S \\ |T|=j}} \nu(T))) - (\sum_{\substack{T \subseteq S \\ |T|=3}} \nu(T) - 3\sum_{\substack{T \subseteq S \\ |T|=4}} \nu(T) + 6\nu(S))$ blocks of $\mathcal{B}$ with exactly one or exactly two points of $\{u, w, x, y, z\}$. Therefore $5r - 10\lambda + 2\sum_{\substack{T \subseteq S \\ |T|=4}} \nu(T) - 5\nu(S)$ is a lower bound on the size of the cutset C. Note that each block that is counted by $\nu(S)$ is counted $\binom{5}{4}$ times within the sum $\sum_{\substack{T \subseteq S \\ |T|=4}} \nu(T)$ and so $2\sum_{\substack{T \subseteq S \\ |T|=4}} \nu(T) \geq 5\nu(S)$. Thus $|C| \geq 5\lambda\left(\frac{v-1}{4}\right) - 10\lambda \geq \frac{v(2\lambda-1)}{5}$ for $v \geq 14$.

Case 1a)ii: Suppose $|\{y, z\} \cap \{y', z'\}| = 1$. Without loss of generality, assume

$y = y'$ and let $U = \{u, w, x, y, z, z'\}$. From the inclusion-exclusion principle there are $6r - 15\lambda + \sum_{j=3}^{5}((-1)^{j-1}\sum_{\substack{T\subseteq U\\|T|=j}}\nu(T))$ blocks of $\mathcal{B}$ containing at least one point of U. Also, there are $\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T)-3\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)+6\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)$ blocks of $\mathcal{B}$ containing at least three points of U. Hence there are $(6r-15\lambda+\sum_{j=3}^{5}((-1)^{j-1}\sum_{\substack{T\subseteq U\\|T|=j}}\nu(T)))-(\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T)-3\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)+6\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T))$ blocks of $\mathcal{B}$ with exactly one or exactly two points of U. Of these blocks, all except those containing exactly one point of $\{z, z'\}$ must be in C. Since exactly $2(r - \lambda)$ blocks contain exactly one point of $\{z, z'\}$, it follows that $(6r - 15\lambda + 2\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T) - 5\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)) - (2r - 2\lambda) = 4r - 13\lambda + 2\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T) - 5\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)$ is a lower bound on the size of the cutset C. Observe that each block that is counted within the sum $\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)$ is counted exactly once within this sum, but is counted $\binom{5}{4}$ times within the sum $\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)$ and so $2\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T) \geq 5\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)$. Thus $|C| \geq 4\lambda\left(\frac{v-1}{4}\right) - 13\lambda \geq \frac{v(2\lambda-1)}{5}$ for $v \geq 24$.

Case 1a)iii: Suppose $|\{y, z\} \cap \{y', z'\}| = 0$ and let $U = \{u, w, x, y, y', z, z'\}$. From the inclusion-exclusion principle there are $7r - 21\lambda + \sum_{j=3}^{5}((-1)^{j-1}\sum_{\substack{T\subseteq U\\|T|=j}}\nu(T))$ blocks of $\mathcal{B}$ containing at least one point of U. Also, there are $\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T) - 3\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T) + 6\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)$ blocks containing at least three points of U. Hence there are $(7r - 21\lambda + \sum_{j=3}^{5}((-1)^{j-1}\sum_{\substack{T\subseteq U\\|T|=j}}\nu(T))) - (\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T) - 3\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T) + 6\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T))$ blocks of $\mathcal{B}$ with exactly one or exactly two points of U. Of these blocks, all except the following must be in C:

1. those which contain the pair yz or the pair $y'z'$.

2. those which have exactly one of y, z, y', z' but none of u, w, x.

There are $2r - 2\lambda$ blocks that contain y or z but not both and there are $2r - 2\lambda$ blocks that contain y' or z' but not both. Hence there are at most $4r - 4\lambda$ exceptional blocks of Type 2 and at most a further 2λ blocks of Type 1, for a total of at most $4r - 2\lambda$ exceptional blocks. It follows that $(7r - 21\lambda + 2\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T) - 5\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)) - (4r - 2\lambda) = 3r - 19\lambda + 2\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T) - 5\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)$ is a lower bound on the size of the cutset C. Once again $2\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T) \geq 5\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)$ and thus $|C| \geq 3\lambda\left(\frac{v-1}{4}\right) - 19\lambda \geq \frac{v(2\lambda-1)}{5}$ for $v \geq 57$.

Case 1b: Assume $V_A \cap V_F = \emptyset$. That is, there is no block $S \in G_{\{1,2\}}(\mathcal{D}) - C$ satisfying $V_A \cap S \neq \emptyset \neq V_F \cap S$. Each block of $\mathcal{B}$ that contains a pair of points wx such that $w \in V_A$ and $x \in V_F$ must belong to C. Clearly there are $|V_A||V_F|$ such pairs of points. Since a block of size five can contain at most six pairs of this kind, and each pair of points occurs in λ blocks, $|C| \geq \frac{\lambda|V_A||V_F|}{6}$. Also, each component of $G_{\{1,2\}}(\mathcal{D}) - C$ has at least one block, so $|V_A| \geq 5$ and $|V_F| \geq 5$. Hence $|C| \geq \min_{5\leq p\leq v-5}\left\{\frac{\lambda p(v-p)}{6}\right\} \geq \frac{5\lambda(v-5)}{6} \geq \frac{v(2\lambda-1)}{5}$ for $v \geq 10$.

Case 2: Assume $V_A \cup V_F \subsetneq V$. Then all blocks containing any point of $V - (V_A \cup V_F)$ must be in C. If there are at least two points in $V - (V_A \cup V_F)$ then

$|C| \geq 2r - \lambda \geq \frac{v(2\lambda-1)}{5}$ for $v \geq 15$.

Now suppose that $|V - (V_A \cup V_F)| = 1$ and let γ be the only element of $V - (V_A \cup V_F)$. So $|V_A \cup V_F| = v - 1$. If $V_A \cap V_F = \emptyset$, then similar to Case 1b, we obtain $|C| \geq \min_{5 \leq p \leq v-6} \left\{ \frac{\lambda p(v-1-p)}{6} \right\} \geq \frac{5\lambda(v-6)}{6} \geq \frac{v(2\lambda-1)}{5}$ for $v \geq 12$.

If $V_A \cap V_F \neq \emptyset$, then similar to Case 1a, there is a block $S = \{u, w, x, y, z\} \in A$ and a block $S' = \{u, w, x, y', z'\} \in F$ having at least three points in common, say u, w and x. Then C includes all blocks of the following types.

1. All those containing γ.

2. All those containing exactly one of u, w, x and none of y, y', z, z', γ.

Clearly, there are r blocks of Type 1. To count the number of blocks of Type 2, we first note that there are at least $3r - 2\binom{3}{2}\lambda$ blocks containing exactly one of the points of $\{u, w, x\}$. At most 15λ blocks may contain a pair of points consisting of a point from $\{u, w, x\}$ and a point from $\{y, y', z, z', \gamma\}$ and so there are at least $(3r - 6\lambda) - 15\lambda$ blocks of Type 2. Therefore $|C| \geq r + (3r - 21\lambda) = 4r - 21\lambda \geq \frac{v(2\lambda-1)}{5}$ for $v \geq 37$.

Therefore $\kappa(G_{\{1,2\}}(\mathcal{D})) \geq \frac{v(2\lambda-1)}{5}$ for all $v \geq 51$ and $\alpha(G_{\{1,2\}}(\mathcal{D})) \leq \kappa(G_{\{1,2\}}(\mathcal{D}))$ in all cases. Hence the Chvátal-Erdős condition holds, and so $G_{\{1,2\}}(\mathcal{D})$ is Hamiltonian.

$\square$

Theorem 3.3. The $\{1,2\}$-block-intersection graph of a $(v,6,\lambda)$-BIBD with $v \geq 167$ and arbitrary λ is Hamiltonian.

Proof. Let $\mathcal{D} = (V, \mathcal{B})$ be a $(v,6,\lambda)$-BIBD and $G_{\{1,2\}}(\mathcal{D}) = (\mathcal{B}, E)$ denote the $\{1,2\}$-block-intersection graph of $\mathcal{D}$. We will show that $G_{\{1,2\}}(\mathcal{D})$ is Hamiltonian by showing that $\alpha(G_{\{1,2\}}(\mathcal{D})) \leq \kappa(G_{\{1,2\}}(\mathcal{D}))$.

First choose an independent set of vertices $I \subset \mathcal{B}$ of $G_{\{1,2\}}(\mathcal{D})$. Then by Lemma 2.2 we have $|I| \leq \frac{v(5\lambda-3)}{12}$. Hence $\alpha(G_{\{1,2\}}(\mathcal{D})) \leq \frac{v(5\lambda-3)}{12}$.

Now let $C \subset \mathcal{B}$ be a cutset of $G_{\{1,2\}}(\mathcal{D})$. We wish to show that $|C| \geq \frac{v(5\lambda-3)}{12}$. Let A be the vertex set of a component of $G_{\{1,2\}}(\mathcal{D}) - C$. Then let $F = \mathcal{B} - (A \cup C)$, $V_A = \bigcup\limits_{\beta \in A} \beta$ and $V_F = \bigcup\limits_{\beta \in F} \beta$.

We now consider two cases.

Case 1: $V_A \cup V_F = V$.

Case 1a: Assume first that $V_A \cap V_F \neq \emptyset$. Then there is a block $S = \{t,u,w,x,y,z\} \in A$ and a block $S' = \{t,u,w,x',y',z'\} \in F$ having at least three points in common, say t, u and w.

Case 1a)i: Suppose $|\{x,y,z\} \cap \{x',y',z'\}| = 3$. Without loss of generality, assume $x = x'$, $y = y'$ and $z = z'$. From the inclusion-exclusion principle there are $6r - 15\lambda + \sum\limits_{j=3}^{6}((-1)^{j-1} \sum\limits_{\substack{T \subseteq S \\ |T|=j}} \nu(T))$ blocks of $\mathcal{B}$ containing at least one point of S. Also, there are $\sum_{\substack{T\subseteq S\\|T|=3}}\nu(T)-3\sum_{\substack{T\subseteq S\\|T|=4}}\nu(T)+6\sum_{\substack{T\subseteq S\\|T|=5}}\nu(T)-10\nu(S)$ blocks of $\mathcal{B}$ containing at least three points of S. Hence there are $(6r-15\lambda+\sum_{j=3}^{6}((-1)^{j-1}\sum_{\substack{T\subseteq S\\|T|=j}}\nu(T)))-(\sum_{\substack{T\subseteq S\\|T|=3}}\nu(T)-3\sum_{\substack{T\subseteq S\\|T|=4}}\nu(T)+6\sum_{\substack{T\subseteq S\\|T|=5}}\nu(T)-10\nu(S))=6r-15\lambda+2\sum_{\substack{T\subseteq S\\|T|=4}}\nu(T)-5\sum_{\substack{T\subseteq S\\|T|=5}}\nu(T)+9\nu(S)$ blocks of $\mathcal{B}$ with exactly one or exactly two points of S. Note that each block β that is counted by $\sum_{\substack{T\subseteq S\\|T|=5}}\nu(T)$ is counted either exactly once (if $|\beta\cap S|=5$) or exactly $\binom{6}{5}$ times (if $|\beta\cap S|=6$). If $|\beta\cap S|=5$, then β is counted $\binom{5}{4}$ times within the sum $\sum_{\substack{T\subseteq S\\|T|=4}}\nu(T)$, and if $|\beta\cap S|=6$, then β is counted $\binom{6}{4}$ times within the sum $\sum_{\substack{T\subseteq S\\|T|=4}}\nu(T)$. Either way β is counted $\binom{|\beta\cap S|}{4}$ times within the sum $\sum_{\substack{T\subseteq S\\|T|=4}}\nu(T)$ and so $2\sum_{\substack{T\subseteq S\\|T|=4}}\nu(T)\geq 5\sum_{\substack{T\subseteq S\\|T|=5}}\nu(T)$. Therefore $6r-15\lambda$ is a lower bound on the size of the cutset C and $|C|\geq 6\lambda\left(\frac{v-1}{5}\right)-15\lambda\geq\frac{v(5\lambda-3)}{12}$ for $v\geq 21$.

Case 1a)ii: Suppose $|\{x,y,z\}\cap\{x',y',z'\}|=2$. Without loss of generality, assume $x=x'$, $y=y'$ and let $U=\{t,u,w,x,y,z,z'\}$. From the inclusion-exclusion principle there are $7r-21\lambda+\sum_{j=3}^{6}((-1)^{j-1}\sum_{\substack{T\subseteq U\\|T|=j}}\nu(T))$ blocks of $\mathcal{B}$ containing at least one point of U. Also, there are $\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T)-3\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)+6\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)-10\sum_{\substack{T\subseteq U\\|T|=6}}\nu(T)$ blocks of $\mathcal{B}$ containing at least three points of U. Hence there are $(7r-21\lambda+\sum_{j=3}^{6}((-1)^{j-1}\sum_{\substack{T\subseteq U\\|T|=j}}\nu(T)))-(\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T)-3\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)+6\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)-10\sum_{\substack{T\subseteq U\\|T|=6}}\nu(T))=7r-21\lambda+2\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)-5\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)+9\sum_{\substack{T\subseteq U\\|T|=6}}\nu(T)$ blocks of $\mathcal{B}$ with exactly one or exactly two points of U. Of these blocks, all except those containing exactly one point of $\{z,z'\}$ must be in C. Since exactly $2(r-\lambda)$ blocks contain exactly one point of $\{z,z'\}$, it follows that $(7r-21\lambda+2\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)-5\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)+9\sum_{\substack{T\subseteq U\\|T|=6}}\nu(T))-(2r-2\lambda)=5r-19\lambda+2\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)-5\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)+9\sum_{\substack{T\subseteq U\\|T|=6}}\nu(T)$ is a lower bound on the size of the cutset C. Note that each block β that is counted by $\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)$ is counted either exactly once (if $|\beta\cap U|=5$) or exactly $\binom{6}{5}$ times (if $|\beta\cap U|=6$). If $|\beta\cap U|=5$, then β is counted $\binom{5}{4}$ times within the sum $\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)$, and if $|\beta\cap U|=6$, then β is counted $\binom{6}{4}$ times within the sum $\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)$. Either way β is counted $\binom{|\beta\cap U|}{4}$ times within the sum $\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)$ and so $2\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)\geq 5\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)$. Thus $|C|\geq 5\lambda\left(\frac{v-1}{5}\right)-19\lambda\geq\frac{v(5\lambda-3)}{12}$ for $v\geq 35$.

Case 1a)iii: Suppose $|\{x,y,z\}\cap\{x',y',z'\}|=1$. Without loss of generality, assume $x=x'$ and let $U=\{t,u,w,x,y,y',z,z'\}$. From the inclusion-exclusion principle there are $8r-28\lambda+\sum_{j=3}^{6}((-1)^{j-1}\sum_{\substack{T\subseteq U\\|T|=j}}\nu(T))$ blocks of $\mathcal{B}$ containing at least one point of U. Also, there are $\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T)-3\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)+6\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)-10\sum_{\substack{T\subseteq U\\|T|=6}}\nu(T)$ blocks of $\mathcal{B}$ containing at least three points of U. Hence there are $(8r-28\lambda+\sum_{j=3}^{6}((-1)^{j-1}\sum_{\substack{T\subseteq U\\|T|=j}}\nu(T)))-(\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T)-3\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)+6\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)-10\sum_{\substack{T\subseteq U\\|T|=6}}\nu(T))$ blocks of $\mathcal{B}$ with exactly one or exactly two points of U. Of these blocks, all except the following must be in C:

1. those which contain the pair yz or the pair $y'z'$.

2. those which have exactly one of y,z,y',z' and none of t,u,w,x.

Similar to Case 1a)iii of Theorem 3.2, there are at most $4r-2\lambda$ of these exceptional blocks, and so it follows that $(8r-28\lambda+2\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)-5\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)+9\sum_{\substack{T\subseteq U\\|T|=6}}\nu(T))-(4r-2\lambda) = 4r-26\lambda+2\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)-5\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)+9\sum_{\substack{T\subseteq U\\|T|=6}}\nu(T)$ is a lower bound on the size of the cutset C. As in previous cases, $2\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T) \geq 5\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)$. Thus $|C| \geq 4\lambda\left(\frac{v-1}{5}\right) - 26\lambda \geq \frac{v(5\lambda-3)}{12}$ for $v \geq 70$.

Case 1a)iv: Suppose $|\{x,y,z\}\cap\{x',y',z'\}| = 0$ and let $U = \{t,u,w,x,x',y,y',z,z'\}$. From the inclusion-exclusion principle there are $9r-36\lambda+\sum_{j=3}^{6}((-1)^{j-1}\sum_{\substack{T\subseteq U\\|T|=j}}\nu(T))$ blocks of $\mathcal{B}$ containing at least one point of U. Also, there are $\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T)-3\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)+6\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)-10\sum_{\substack{T\subseteq U\\|T|=6}}\nu(T)$ blocks of $\mathcal{B}$ containing at least three points of U. Hence there are $(9r-36\lambda+\sum_{j=3}^{6}((-1)^{j-1}\sum_{\substack{T\subseteq U\\|T|=j}}\nu(T)))-(\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T)-3\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)+6\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T)-10\sum_{\substack{T\subseteq U\\|T|=6}}\nu(T))$ blocks of $\mathcal{B}$ with exactly one or exactly two points of U. Of these blocks, all except the following must be in C:

1. those which contain one of the pairs: xy, xz, yz, $x'y'$, $x'z'$, or $y'z'$.

2. those which have exactly one of x, x', y, y', z, z' and none of t, u, w.

There are at most $3r-6\lambda$ blocks that contain x, y or z but not both and there are at most $3r-6\lambda$ blocks that contain x', y' or z' but not both. Hence there are at most $6r-12\lambda$ exceptional blocks of Type 2 and at most a further 6λ blocks of

Type 1, for a total of at most $6r - 6\lambda$ exceptional blocks. It follows that $(9r - 36\lambda + 2\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T) - 5\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T) + 9\sum_{\substack{T\subseteq U\\|T|=6}}\nu(T)) - (6r - 6\lambda)$ is a lower bound on the size of the cutset C. Once again, $2\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T) - 5\sum_{\substack{T\subseteq U\\|T|=5}}\nu(T) \geq 0$. Thus $|C| \geq 3\lambda\left(\frac{v-1}{5}\right) - 30\lambda \geq \frac{v(5\lambda-3)}{12}$ for $v \geq 167$.

Case 1b: Assume $V_A \cap V_F = \emptyset$. That is, there is no block $S \in G_{\{1,2\}}(\mathcal{D}) - C$ satisfying $V_A \cap S \neq \emptyset \neq V_F \cap S$. Without loss of generality, assume that A is a smallest component of $G_{\{1,2\}}(\mathcal{D}) - C$.

Case 1b)i: Suppose $|A| = 1$. Without loss of generality, let A contain the block $\{t, u, w, x, y, z\}$. Thus, $V_A = \{t, u, w, x, y, z\}$, $|V_A| = 6$ and $|V_F| = v - 6$. Then all $6(v-6)\lambda$ mixed pairs (those pairs that contain a point from both V_A and V_F) are in blocks of C. The blocks of C with mixed pairs are of the following types.

Type 1. $AAAAAD$: 5 mixed pairs and 10 pure A pairs.

Type 2. $AAAADD$: 8 mixed pairs and 6 pure A pairs.

Type 3. $AAADDD$: 9 mixed pairs and 3 pure A pairs.

Type 4. $AADDDD$: 8 mixed pairs and 1 pure A pair.

Type 5. $ADDDDD$: 5 mixed pairs and 0 pure A pairs.

The overall number of pure A pairs (those that contain two points of V_A) is $\lambda\binom{6}{2} = 15\lambda$. The number of Type 3 blocks is at most $\frac{\binom{6}{2}(\lambda-1)}{3} = 5\lambda - 5$, in which there are at most $9(5\lambda - 5)$ mixed pairs. Hence $|C| \geq \min_{0\leq i\leq 5\lambda-5}\left\{i + \frac{6\lambda(v-6)-9i}{8}\right\} \geq$

$\frac{6\lambda(v-6)-(5\lambda-5)}{8} \geq \frac{v(5\lambda-3)}{12}$ for $v \geq 16$.

Case 1b)ii: If $|A| \geq 2$, then $|V_A| \geq 10$ and $|V_F| \geq 10$. Now, the number of mixed pairs is at least $\min_{10 \leq p \leq v-10} \{p(v-p)\lambda\}$. Hence $|C| \geq \frac{10(v-10)\lambda}{9} \geq \frac{v(5\lambda-3)}{12}$ for $v \geq 16$.

Case 2: Assume $V_A \cup V_F \subsetneq V$. Then all blocks containing any element of $V - (V_A \cup V_F)$ must be in C.

Case 2a: Assume $|V - (V_A \cup V_F)| = 1$ and let γ be the only element of $V - (V_A \cup V_F)$. So $|V_A \cup V_F| = v - 1$.

Case 2a)i: Suppose $V_A \cap V_F = \emptyset$. If $|A| = 1$, then similar to Case 1b, $|C| \geq \min_{0 \leq i \leq 5\lambda-5} \left\{ i + \frac{6\lambda(v-7)-9i}{8} \right\} \geq \frac{6\lambda(v-7)-(5\lambda-5)}{8} \geq \frac{v(5\lambda-3)}{12}$ for $v \geq 18$.

If $|A| \geq 2$, then $|V_A| \geq 10$ and $|V_F| \geq 10$. Now the number of mixed pairs is at least $\min_{10 \leq p \leq v-11} \{p(v-p-1)\lambda\}$. Hence $|C| \geq \frac{10(v-11)\lambda}{9} \geq \frac{v(5\lambda-3)}{12}$ for $v \geq 18$.

Case 2a)ii: Suppose $V_A \cap V_F \neq \emptyset$. Then similar to Case 1a there is a block $S = \{t, u, w, x, y, z\} \in A$, and a block $S' = \{t, u, w, x', y', z'\} \in F$ having at least three points in common, say t, u and w. Then C includes all blocks of the following types.

1. All those containing γ.

2. All those containing exactly one of t, u, w and none of $x, x', y, y', z, z', \gamma$.

There are r blocks of Type 1 and at least $3r - 27\lambda$ blocks of Type 2. Hence $|C| \geq r + (3r - 27\lambda) = 4r - 27\lambda \geq \frac{v(5\lambda-3)}{12}$ for $v \geq 73$.

Case 2b: Now assume $|V-(V_A\cup V_F)|=2$ and let $\{\gamma_1,\gamma_2\}=V-(V_A\cup V_F)$.

Case 2b)i: Suppose $V_A\cap V_F=\emptyset$. If $|A|=1$, $|V_A|=6$, $|V_F|=v-8$, and every mixed pair tu such that $t\in V_A\cup\{\gamma_1,\gamma_2\}$ and $u\in V_F$ must be in C. The number of such pairs is $\lambda(8)(v-8)$. Hence $|C|\geq\frac{8\lambda(v-8)}{9}\geq\frac{v(5\lambda-3)}{12}$ for $v\geq 16$.

If $|A|\geq 2$, then $|V_A|\geq 10$, $|V_F|\geq 10$ and every mixed pair tu such that $t\in V_A\cup\{\gamma_1,\gamma_2\}$ and $u\in V_F$ must be in C. Now the number of mixed pairs is at least $\min\limits_{10\leq p\leq v-12}\{(p+2)(v-p-2)\lambda\}$. Hence $|C|\geq\frac{12(v-12)\lambda}{9}\geq\frac{v(5\lambda-3)}{12}$ for $v\geq 18$.

Case 2b)ii: Suppose $V_A\cap V_F\neq\emptyset$. Then similar to Case 1a there is a block $S=\{t,u,w,x,y,z\}\in A$ and a block $S'=\{t,u,w,x',y',z'\}\in F$ having at least three points in common, say t,u, and w. The cutset C must include all blocks of the following types.

1. All those containing γ_1,γ_2 or both.

2. All those containing exactly one of $\{t,u,w\}$ and none of $\{x,x',y,y',z,z',\gamma_1,\gamma_2\}$.

There are $2r-\lambda$ blocks of Type 1 and at least $3r-30\lambda$ blocks of Type 2. Hence $|C|\geq(2r-\lambda)+(3r-30\lambda)=5r-31\lambda\geq\frac{v(5\lambda-3)}{12}$ for $v\geq 55$.

Case 2c: Finally suppose $|V-(V_A\cup V_F)|\geq 3$ and let $S\subseteq V-(V_A\cup V_F)$ such that $|S|=3$. Then $|C|\geq 3r-3\lambda+\sum\limits_{\substack{T\subset S\\|T|=3}}\nu(T)\geq 3r-3\lambda\geq\frac{v(5\lambda-3)}{12}$ for $v\geq 20$.

Therefore $\kappa(G_{\{1,2\}}(\mathcal{D}))\geq\frac{v(5\lambda-3)}{12}$ for all $v\geq 167$ and $\alpha(G_{\{1,2\}}(\mathcal{D}))\leq\kappa(G_{\{1,2\}}(\mathcal{D}))$

in all cases. Hence the Chvátal-Erdős condition holds, and so $G_{\{1,2\}}(\mathcal{D})$ is Hamiltonian.

$\square$

Chapter 4

The 1-Block-Intersection Graph

In this chapter we will extend the idea in [11] to BIBDs of block size four by showing that the 1-block-intersection graph of any $(v, 4, \lambda)$-BIBD with $v \geq 136$ and arbitrary λ is Hamiltonian.

Theorem 4.1. The 1-block-intersection graph of any $(v, 4, \lambda)$-BIBD with $v \geq 136$ and arbitrary λ is Hamiltonian.

Proof. Let $\mathcal{D} = (V, \mathcal{B})$ be a $(v, 4, \lambda)$-BIBD and $G_1(\mathcal{D}) = (\mathcal{B}, E)$ denote the 1-block-intersection graph of $\mathcal{D}$. We will show that $G_1(\mathcal{D})$ is Hamiltonian by showing that $\alpha(G_1(\mathcal{D})) \leq \kappa(G_1(\mathcal{D}))$.

First choose an independent set of vertices $I \subset \mathcal{B}$ of $G_1(\mathcal{D})$. Then by Lemma 2.4 we have $|I| \leq \frac{v(3\lambda-2)}{4}$. Hence $\alpha(G_1(\mathcal{D})) \leq \frac{v(3\lambda-2)}{4}$.

Now let $C \subset \mathcal{B}$ be a cutset of $G_1(\mathcal{D})$. We wish to show that $|C| \geq \frac{v(3\lambda-2)}{4}$. Let A be the vertex set of a smallest component of $G_1(\mathcal{D}) - C$. Then let $F = \mathcal{B} - (A \cup C)$, $V_A = \bigcup_{\beta \in A} \beta$ and $V_F = \bigcup_{\beta \in F} \beta$.

If $|A| = 1$, then from the inclusion-exclusion principle there are $4r - 6\lambda + \sum_{\substack{T \subseteq V_A \\ |T|=3}} \nu(T) - \nu(V_A)$ blocks of $\mathcal{B}$ having at least one point in common with V_A. Also, there are $6\lambda - 2\sum_{\substack{T \subseteq V_A \\ |T|=3}} \nu(T) + 3\nu(V_A)$ blocks of $\mathcal{B}$ having at least two points in common with V_A. Hence there are $(4r - 6\lambda + \sum_{\substack{T \subseteq V_A \\ |T|=3}} \nu(T) - \nu(V_A)) - (6\lambda - 2\sum_{\substack{T \subseteq V_A \\ |T|=3}} \nu(T) + 3\nu(V_A))$ blocks of $\mathcal{B}$ having exactly one point in common with V_A. Note that each block that is counted by $\nu(V_A)$ is counted $\binom{4}{3}$ times within the sum $\sum_{\substack{T \subseteq V_A \\ |T|=3}} \nu(T)$ and so $3\sum_{\substack{T \subseteq V_A \\ |T|=3}} \nu(T) \geq 4\nu(V_A)$. Thus $|C| \geq 4r - 12\lambda + 3\sum_{\substack{T \subseteq V_A \\ |T|=3}} \nu(T) - 4\nu(V_A) \geq 4\lambda\left(\frac{v-1}{3}\right) - 12\lambda \geq \frac{v(3\lambda-2)}{4}$ for $v \geq 23$.

Henceforth we assume $|A| \geq 2$ and therefore $|V_A| \geq 7$ and $|V_F| \geq 7$.

We now consider two cases.

Case 1: $V_A \cup V_F = V$.

Case 1a: Assume first that $V_A \cap V_F \neq \emptyset$. Then there is a block $S = \{w, x, y, z\} \in A$ and a block $S' = \{w, x, y', z'\} \in F$ having at least two points in common, say w and x.

Case 1a)i: Suppose $|\{y, z\} \cap \{y', z'\}| = 2$. Without loss of generality, assume $y = y'$ and $z = z'$. From the inclusion-exclusion principle there are $4r - 6\lambda +$

$\sum_{\substack{T\subseteq S\\|T|=3}}\nu(T) - \nu(S)$ blocks of $\mathcal{B}$ having at least one point in common with S. Also, there are $6\lambda - 2\sum_{\substack{T\subseteq S\\|T|=3}}\nu(T) + 3\nu(S)$ blocks of $\mathcal{B}$ having at least two points in common with S. Hence there are $(4r - 6\lambda + \sum_{\substack{T\subseteq S\\|T|=3}}\nu(T) - \nu(S)) - (6\lambda - 2\sum_{\substack{T\subseteq S\\|T|=3}}\nu(T) + 3\nu(S))$ blocks of $\mathcal{B}$ having exactly one point in common with S. Note that each block that is counted by $\nu(S)$ is counted $\binom{4}{3}$ times within the sum $\sum_{\substack{T\subseteq S\\|T|=3}}\nu(T)$ and so $3\sum_{\substack{T\subseteq S\\|T|=3}}\nu(T) \geq 4\nu(S)$. Thus $|C| \geq 4r - 12\lambda + 3\sum_{\substack{T\subseteq S\\|T|=3}}\nu(T) - 4\nu(S) \geq 4\lambda\left(\frac{v-1}{3}\right) - 12\lambda \geq \frac{v(3\lambda-2)}{4}$ for $v \geq 23$.

Case 1a)ii: Suppose $|\{y,z\} \cap \{y',z'\}| = 1$. Without loss of generality, assume $y = y'$ and let $U = \{w,x,y,z,z'\}$. From the inclusion-exclusion principle there are $5r - 10\lambda + \sum_{j=3}^{4}((-1)^{j-1}\sum_{\substack{T\subseteq U\\|T|=j}}\nu(T))$ blocks of $\mathcal{B}$ having at least one point of U. Also, there are $10\lambda - 2\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T) + 3\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)$ blocks of $\mathcal{B}$ containing at least two points of U. Hence there are $(5r - 10\lambda + \sum_{j=3}^{4}((-1)^{j-1}\sum_{\substack{T\subseteq U\\|T|=j}}\nu(T))) - (10\lambda - 2\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T) + 3\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T))$ blocks of $\mathcal{B}$ containing exactly one point of U. Of these blocks, all except those containing exactly one point of $\{z,z'\}$ must be in C. Since exactly $2(r-\lambda)$ blocks contain exactly one point of $\{z,z'\}$, it follows that $(5r - 20\lambda + 3\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T) - 4\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)) - (2r - 2\lambda)$ is a lower bound on the size of the cutset C. Note that each block that is counted by $\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)$ is counted $\binom{4}{3}$ times within the sum $\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T)$ and so $3\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T) \geq 4\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T)$. Thus $|C| \geq 3r - 18\lambda + 3\sum_{\substack{T\subseteq U\\|T|=3}}\nu(T) - 4\sum_{\substack{T\subseteq U\\|T|=4}}\nu(T) \geq 3\lambda\left(\frac{v-1}{3}\right) - 18\lambda \geq \frac{v(3\lambda-2)}{4}$ for $v \geq 76$.

Case 1a)iii: Suppose $|\{y,z\} \cap \{y',z'\}| = 0$ and let $U = \{w,x,y,y',z,z'\}$.

Initially we suppose that $|V_A \cap V_F| = 2$ and so $V_A \cap V_F = \{w, x\}$. Then there are $(|V_A|-2)(|V_F|-2)$ pairs of points containing one element from $V_A - (V_A \cap V_F)$ and the other from $V_F - (V_A \cap V_F)$. Since a block of size four can only contain at most four pairs of this kind and each pair occurs in λ blocks, we have $|C| \geq \frac{\lambda(|V_A|-2)(|V_F|-2)}{4}$. Recall that $|A| \geq 2$, $|V_A| \geq 7$ and $|V_F| \geq 7$. Hence $|C| \geq \min_{7 \leq p \leq v-5} \left\{ \frac{\lambda(p-2)(v-p)}{4} \right\} \geq \frac{5\lambda(v-7)}{4} \geq \frac{v(3\lambda-2)}{4}$ for $v \geq 18$.

Now suppose that $|V_A \cap V_F| \geq 3$. Then in addition to w and x there exists a third point $t \in V_A \cap V_F$ and so there exists a pair of blocks $T \in A$ and $T' \in F$ with t in common. Let $D = S \cup S' \cup T \cup T'$. If neither T nor T' is the same block as S or S' then $\{S, S', T, T'\}$ is a set of four blocks and $6 \leq |D| \leq 12$. The number of blocks containing exactly one of w, x, t and none of $D - \{w, x, t\}$ is at least $3r - \binom{3}{2}(2\lambda) - 3(|D|-3)\lambda$. So $|C| \geq 3r + 3\lambda - 3|D|\lambda \geq 3\lambda\left(\frac{v-1}{3}\right) - 33\lambda \geq \frac{v(3\lambda-2)}{4}$ for $v \geq 136$.

If T is the same block as S, but T' is not S' then there are only three blocks in play. Since $|D|$ can then be at most eight, it follows that $|C| \geq 3r - 21\lambda \geq \frac{v(3\lambda-2)}{4}$ for $v \geq 88$.

If T is the same as S and T' is the same as S' then we are either in Case 1a)ii or Case 1a)i.

Case 1b: Assume $V_A \cap V_F = \emptyset$. That is, there is no block $S \in G_1(\mathcal{D}) - C$

satisfying $V_A \cap S \neq \emptyset \neq V_F \cap S$. Each block of $\mathcal{B}$ that contains a pair of points wx such that $w \in V_A$ and $x \in V_F$ must belong to C. Clearly, there are $|V_A||V_F|$ of these pairs. Since a block of size four can only contain at most four pairs of this kind and each pair of points occurs in λ blocks, we have $|C| \geq \frac{\lambda|V_A||V_F|}{4}$. Also, each component of $G_1(\mathcal{D}) - C$ has at least two blocks, so $|V_A| \geq 7$ and $|V_F| \geq 7$. Hence $|C| \geq \min_{7 \leq p \leq v-7} \left\{\frac{\lambda p(v-p)}{4}\right\} \geq \frac{7\lambda(v-7)}{4} \geq \frac{v(3\lambda-2)}{4}$ for $v \geq 13$.

Case 2: Assume $V_A \cup V_F \subsetneq V$. Then all blocks containing any element of $V - (V_A \cup V_F)$ must be in C.

Case 2a: Assume $|V - (V_A \cup V_F)| = 1$ and let γ be the only element of $V - (V_A \cup V_F)$. So $|V_A \cup V_F| = v - 1$.

Case 2a)i: If $V_A \cap V_F = \emptyset$, then similar to Case 1b we obtain $|C| \geq \min_{7 \leq p \leq v-8} \left\{\frac{\lambda p(v-1-p)}{4}\right\} \geq \frac{7\lambda(v-8)}{4} \geq \frac{v(3\lambda-2)}{4}$ for $v \geq 14$.

Case 2a)ii: If $V_A \cap V_F \neq \emptyset$, then similar to Case 1a there is a block $S = \{w, x, y, z\} \in A$, and a block $S' = \{w, x, y', z'\} \in F$ having at least two points in common, say w and x. Then C includes all blocks of the following types.

1. All those containing γ.

2. All those containing exactly one of w, x and none of y, y', z, z', γ.

There are r blocks of Type 1 and at least $2r - 12\lambda$ blocks of Type 2. Hence $|C| \geq r + (2r - 12\lambda) = 3r - 12\lambda \geq \frac{v(3\lambda-2)}{4}$ for $v \geq 52$.

Case 2b: Now assume $|V-(V_A \cup V_F)|=2$. Let $\{\gamma_1,\gamma_2\} = V-(V_A \cup V_F)$.

Case 2b)i: Assume $V_A \cap V_F = \emptyset$. Since $|A| \geq 2$, then $|V_A| \geq 7$ and $|V_F| \geq 7$. Every mixed pair wx such that $w \in V_A \cup \{\gamma_1,\gamma_2\}$ and $x \in V_F$ must be in C. Now the number of mixed pairs is at least $\min\limits_{7 \leq p \leq v-9} \{(p+2)(v-p-2)\lambda\}$. Hence $|C| \geq \min\limits_{7 \leq p \leq v-9} \left\{\frac{(p+2)(v-p-2)\lambda}{4}\right\} \geq \frac{9(v-9)\lambda}{4} \geq \frac{v(3\lambda-2)}{4}$ for $v \geq 14$.

Case 2b)ii: If $V_A \cap V_F \neq \emptyset$, then similar to Case 1a there is a block $S = \{w,x,y,z\} \in A$ and a block $S' = \{w,x,y',z'\} \in F$ having at least two points in common, say w and x. Then C includes all blocks of the following types.

1. All those containing γ_1, γ_2 or both.

2. All those containing exactly one of w, x and none of $y, y', z, z', \gamma_1, \gamma_2$.

There are $2r-\lambda$ blocks of Type 1 and at least $2r-14\lambda$ blocks of Type 2. Hence $|C| \geq (2r-\lambda)+(2r-14\lambda) = 4r-15\lambda \geq \frac{v(3\lambda-2)}{4}$ for $v \geq 28$.

Case 2c: Assume $|V-(V_A \cup V_F)| \geq 3$ and let $S \subseteq V-(V_A \cup V_F)$ such that $|S|=3$. Then $|C| \geq 3r-3\lambda+\sum\limits_{\substack{T \subseteq S \\ |T|=3}} \nu(T) \geq 3\lambda\left(\frac{v-1}{3}\right)-3\lambda \geq \frac{v(3\lambda-2)}{4}$ for $v \geq 16$.

Therefore $\kappa(G_1(\mathcal{D})) \geq \frac{v(3\lambda-2)}{4}$ for all $v \geq 136$ and $\alpha(G_1(\mathcal{D})) \leq \kappa(G_1(\mathcal{D}))$ in all cases. Hence by the Chvátal-Erdős condition holds, and so $G_1(\mathcal{D})$ is Hamiltonian.

□

Chapter 5

Summary and Some Open Problems

In this thesis we have given numerous definitions and examples that deal with graph theory and design theory, several lemmata that deal with bounding the size of independent sets of vertices of block-intersection graphs and we have shown that

(i) the $\{1,2\}$-block-intersection graph for any $(v, 4, \lambda)$-BIBD with $v \geq 11$ and arbitrary λ is Hamiltonian,

(ii) the $\{1,2\}$-block-intersection graph for any $(v, 5, \lambda)$-BIBD with $v \geq 57$ and arbitrary λ is Hamiltonian,

(iii) the $\{1,2\}$-block-intersection graph for any $(v, 6, \lambda)$-BIBD with $v \geq 167$ and arbitrary λ is Hamiltonian,

(iv) the 1-block-intersection graph for any $(v, 4, \lambda)$-BIBD with $v \geq 136$ and arbitrary λ is Hamiltonian.

Some open problems are showing that

1. the $\{1, 2\}$-block-intersection graph for any (v, k, λ)-BIBD with $k \geq 7$ and arbitrary λ is Hamiltonian or else find a suitable counter example,

2. the 1-block-intersection graph for any (v, k, λ)-BIBD with $k \geq 5$ and arbitrary λ is Hamiltonian or else find a suitable counter example.

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